Certain associative algebras similar to $U(sl_2)$ and Zhu's algebra $A(V_L)$

Chongying Dong[1], Haisheng Li and Geoffrey Mason[2]
Department of Mathematics, University of California, Santa Cruz, CA 95064

Abstract

It is proved that Zhu's algebra for vertex operator algebra associated to a positive-definite even lattice of rank one is a finite-dimensional semiprimitive quotient algebra of certain associative algebra introduced by Smith. Zhu's algebra for vertex operator algebra associated to any positive-definite even lattice is also calculated and is related to a generalization of Smith's algebra.

1 Introduction

The recently developed vertex operator algebra theory encopes Lie algebras (both finite-dimensional simple Lie algebras and infinite-dimensional affine Lie algebras), groups (Lie groups and finite groups), codes and lattices. Many important algebras have appeared to be closely related to certain vertex operator algebras. For instance, the Griess algebra is a substructure of Frenkel, Lepowsky and Muerman's Moonshine vertex operator algebra $V^\natural$ whose full symmetry group is the Monster group. Our naive purpose in this paper is to relate a new interesting class of associative algebras appeared in a different field to certain vertex operator algebras.

In [S], Smith studied an interesting class of associative algebras $R(f)$ parameterized by a polynomial $f(x)$. Briefly, for any polynomial $f(x)$, Smith defined an associative algebra $R(f)$ with three generators A, B, H with defining relations $HA - AH = A, HF - FH = -F, AB - BA = f(H)$. As one of the main results in [S], it was shown that for some $f(x)$, $R(f)$ behaves much like $U(sl_2)$ in terms of the complete reducibility of any finite-dimensional $R(f)$-modules. Following [BB], Hodges and Smith established an equivalence between the category of $R(f)$-modules and the category of sheaves of left $\mathcal{D}$-modules in [H] and [HS] where $\mathcal{D}$ is a sheaf of rings on a siutable finite topological space.

In this paper we find a relation between algebra $R(f)$ and vertex operator algebras V_L associated to positive definite even lattices L of rank 1 by realizing Zhu algebras $A(V_L)$ as certain semisimple quotients of $R(f)$ for certain f. This relation suggests us to study a generalization of $R(f)$ associated to any positive definite even lattice which leads to the calculation of $A(V_L)$ in general. In fact we characterize $A(V_L)$ by generators and realtions. It is expected that the realtion between generalization of $R(f)$ and $A(V_L)$ will help us to understand the vertex operator algebra structure of V_L in terms of $\mathcal{D}$-modules on algebraic curves (see [BD] and [HL]).

[1]Supported by NSF grant DMS-9303374 and a research grant from the Committee on Research, UC Santa Cruz.

[2]Supported by NSF grant DMS-9401272 and a research grant from the Committee on Research, UC Santa Cruz.

In [Z], an associative algebra $A(V)$ was introduced for any vertex operator algebra V so that there is a 1-1 correspondence between the set of equivalence classes of irreducible V-modules and the set of equivalence classes of $A(V)$-modules. If V is the irreducible highest weight $\hat{\mathfrak{g}}$-module of level ℓ with lowest weight 0 where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra (cf. [FZ], [L1]), Zhu's algebra is isomorphic to $U(\mathfrak{g})$ if ℓ is generic. If ℓ is a positive integer, Zhu's algebra is isomorphic to a semiprimitive quotient algebra of $U(\mathfrak{g})$ [FZ]. On the other hand, from FKS construction [FLM] of basic modules for affine Lie algebras of types A, D, E, $L(1,0) = V_L$, where L is the root lattice of $\mathfrak{g}$. So Zhu's algebra $A(V_L)$ for an arbitrary positive definite even lattice L is very "close" to the universal enveloping algebra of a semisimple Lie algebra in some sense. It is natural to expect certain $\mathcal{D}$-modules entering the picture in the spirit of [BB].

Here is the precise structure of $A(V_L)$ if $L = \mathbb{Z}\alpha$ be a rank-one lattice with $\langle\alpha,\alpha\rangle = 2k$ for some positive integer k. $A(V_L)$ in this case is a semiprimitive quotient ring of $R(f_k)$ with

$$f_k(x) = \frac{2k}{(2k-1)!}x(4k^2x^2-1)(4k^2x^2-4)\cdots(4k^2x^2-(k-1)^2).$$

In the case $\langle\alpha,\alpha\rangle = 2$ $A(V_L)$ has been computed previously in [Lu].

The paper is organized as follows. In Section 2 we study the algebra $R(f_k)$ and prove that a certain quotient $\bar{R}_k$ of $R(f_k)$ is a semi-simple algebra whose all irreducible representations are given explicitly. A generalization $\bar{A}(L)$ of $R(f)$ associated to an positive definite lattice L is also investigated. We establish the semisimplicity of $\bar{A}(L)$ and construct all its irrducible modules. In Section 3 we prove that $\bar{R}_k$ and $A(V_{\mathbb{Z}\alpha})$ are isomorphic if $\langle\alpha,\alpha\rangle = 2k$ and that $\bar{A}(L)$ and $A(V_L)$ are isomorphic for arbitrary positive definite even lattice L.

2 Certain associative algebras similar to $U(sl_2)$

Let $g(x) \in \mathbb{C}[x]$ be any polynomial in x. The associative algebra $R(g)$ [S] is generated by $\{A, B, H\}$ subject to relations:

$$HA - AH = A, \quad HB - BH = -B, \quad AB - BA = g(H).$$

In fact, $R(g)$ is a $\mathbb{Z}$-graded algebra with $\deg A = 1 = -\deg B$ and $\deg H = 0$, and $A(g)$ has a basis $\{B^m H^n A^k | m, n, k \in \mathbb{Z}_+\}$.

Let P be the subalgebra of $R(g)$ generated by A and H. Then $AP = \{Aa | a \in P\}$ is a two-sided ideal of P and $P = AP \oplus \mathbb{C}[H]$. For any complex number λ, let $\mathbb{C}v_\lambda$ be the 1-dimensional P-module such that $APv_\lambda = 0, Hv_\lambda = \lambda v_\lambda$. Define the Verma $R(g)$-module as follows:

$$V(\lambda) = R(g) \otimes_P \mathbb{C}v_\lambda.$$

Then $V(\lambda)$ has a unique maximal proper submodule. Denote by $L(\lambda)$ the irreducible quotient module of $V(\lambda)$.

Let $u(x)$ be a polynomial of degree $n+1$ and take $g = \frac{1}{2}(u(x+1) - u(x))$. Similar to $U(sl_2)$, $R(g)$ has a central element

$$\Omega = AB + BA + \frac{1}{2}\left(u(H+1) + u(H)\right)$$

such that the center of $R(g)$ is isomorphic to $\mathbb{C}[\Omega]$. Furthermore, Ω acts on $V(\lambda)$ as a scalar $u(\lambda+1)$.

The following two propositions can be found in [S].

Proposition 2.1 *(a) The finite-dimensional simple $R(g)$-modules are precisely the modules $L(\lambda) = V(\lambda)/B^j v_\lambda$ where $j \in \mathbb{N}$ is minimal such that $u(\lambda+1) = u(\lambda+1-j)$.*

(b)The number of simple modules of dimension j equals

$$|\{\lambda \in \mathbb{C} | u(\lambda+1) = u(\lambda+1-j), \text{ and } j \text{ is the least such element of } \mathbb{N}\}|$$

which is less than or equal to $\deg u - 1 = \deg g$.

Proposition 2.2 *Suppose that for each $j \in \mathbb{N}$, there are precisely* $\deg g$ *simple modules of dimension j. Then every finite-dimensional $R(g)$-module is semisimple.*

Set $h_j(x) = g(x) + g(x-1) + \cdots + g(x-j)$ for $j \in \mathbb{Z}_+$. Observe that $h_j(x) = \frac{1}{2}(u(x) - u(x-j))$. The next corollary is implicit in [S].

Corollary 2.3 *If each root of $h_j(x) = 0$ for $j \in \mathbb{Z}_+$ has multiplicity* 1 *and any two equations $h_i(x) = 0, h_j(x) = 0$ do not have common solutions, then every finite-dimensional $R(g)$-module is semisimple.*

Proof. Since $\deg h_{j-1}(x) = \deg(u(x+1) - u(x+1-j)) = \deg g(x)$, $u(x+1) = u(x+1-j)$ has exactly $\deg g$ solutions for fixed $j \in \mathbb{N}$. By the assumption, $u(x+1) = u(x+1-j)$ and $u(x+1) = u(x+1-i)$ have no common solutions if $i \neq j$. Thus for each solution λ of $u(x+1) = u(x+1-j)$, j is the minumum such that $u(\lambda+1) = u(\lambda+1-j)$. By Proposition 2.1 $R(g)$ has exactly $\deg g$ simple modules of dimension j. Use Proposition 2.2 to finish the proof. $\square$

Similar to the complete reducibility of an integrable module for a semisimple Lie algebra (cf. [K]) we have

Corollary 2.4 *If each root of $h_j(x) = 0$ for $j \in \mathbb{Z}_+$ has multiplicity* 1 *and any two equations $h_i(x) = 0, h_j(x) = 0$ do not have common solutions, then every $R(g)$-module on which A and B are locally nilpotent is semisimple.*

Proof. We first prove that any nonzero $R(g)$-module M on which A and B are locally nilpotent contains a finite-dimensional simple $R(g)$-module V.

Set $M^0 = \{u \in M | Au = 0\}$. It is clear that $HM^0 \subseteq M^0$. Since A is locally nilpotent on M, $M^0 \neq 0$. Let $0 \neq v \in M^0$. Then there exists a nonnegative integer r such that $B^r u \neq 0$ and $B^{r+1}v = 0$. Since

$$\begin{aligned}
& 2^{r+1}A^{r+1}B^{r+1} \\
= \; & (\Omega - u(H))(\Omega - u(H-1)) \cdots (\Omega - u(H-r)) \\
= \; & (2BA + u(H+1) - u(H))(2BA + u(H+1) - u(H-1)) \cdots \\
& \cdots (2BA + u(H+1) - u(H-r))
\end{aligned}$$

(cf. [DLM3] or [S]), we obtain

$$(u(H+1) - u(H))(u(H+1) - u(H-1)) \cdots (u(H+1) - u(H-r))v = 0.$$

Consequently, there is a $\lambda \in \mathbb{C}$ and a $0 \neq u \in M^0$ such that $Hu = \lambda u$. Then u generates a highest weight $R(g)$-module V. By Corollary 2.3, V is simple.

Now let W be the sum of all finite-dimensional simple $R(g)$-submodules of M. Then it suffices to prove $M = W$. Suppose $W \neq M$. Then $\bar{M} = M/W$ is a nonzero $R(g)$-module which has a finite-dimensional simple submodule M^1/W where M^1 is a submodule of M. Let $u \in M^1$ such that $u + W$ is a nonzero highest weight vector. Then $R(g)u$ is contained in a finite-dimensional $R(g)$-submodule L of M^1. It follows from Corollary 2.3 that L is a direct sum of finite-dimensional simple $R(g)$-modules so that u is contained in W. It is a contradiction. The proof is complete. $\square$

Smith [S] gave an instructive example for $g(x) = (x+1)^{n+1} - x^{n+1}$. Motivated by vertex operator algebras associated to positive definite even lattices of rank one we consider the associative algebras $R_k = R(g_k)$ for any positive integer k where

$$g_k(x) = \frac{1}{(2k-1)!} 2kx(4k^2x^2 - 1)(4k^2x^2 - 4) \cdots (4k^2x^2 - (k-1)^2).$$

Then R_k is generated by $\{A, B, H\}$ subject to relations:

$$\begin{aligned}
& HA - AH = A, \quad HB - BH = -B, \\
& AB - BA = \frac{1}{(2k-1)!} 2kH(4k^2H^2 - 1)(4k^2H^2 - 4) \cdots (4k^2H^2 - (k-1)^2).
\end{aligned}$$

Notice that $R_1 = U(sl_2)$. If $k = 2$, the corresponding polynomial $u_2(x) = \frac{16}{3}\left(x - \frac{1}{2}\right)^4 - \frac{10}{3}\left(x - \frac{1}{2}\right)^2$. For any positive integer k it may be possible to prove that any finite-dimensional R_k-module is completely reducible. But in this paper we only prove the result for $k = 2$.

Lemma 2.5 *Every R_2-module M on which A and B are locally nilpotent is semisimple.*

Proof. Since $g_2(x) = \frac{2}{3}x(16x^2-1) = \frac{32}{3}x^3 - \frac{2}{3}x$, for any $r \in \mathbb{Z}_+$, we have

$$\begin{aligned} & g_2(x)+g_2(x-1)+\cdots+g_2(x-r) \\ = & \frac{32}{3}\left(x^3+(x-1)^3+\cdots+(x-r)^3\right) - \frac{2}{3}\left(x+(x-1)+\cdots+(x-r)\right) \\ = & \frac{(r+1)}{3}(2x-r)\left(16x^2-16rx+8r^2+8r-1\right). \end{aligned}$$

Notice that $16x^2-16rx+8r^2+8r-1=0$ has two distinct noreal solutions. If λ satisfies two equations $16x^2-16rx+8r^2+8r-1=0$ and $16x^2-16sx+8s^2+8s-1=0$ for $r>s>0$, then λ satisfies the difference equation $16(r-s)x-8(r^2-s^2)-8(r-s)=0$. Consequently, λ is rational. It is a contradiction. Thus any two of the equations $g_2(x)+g_2(x-1)+\cdots+g_2(x-j)=0$ for $j=0,1,\cdots$ do not have common solutions. The result follows from Corollary 2.4 immediately. $\square$

For a fixed positive integer k, we define $\bar{R}_k$ to be the quotient algebra of R_k modulo the two-sided ideal generated by $(1-2H)A$. Next, we shall prove that $\bar{R}_k$ is a semisimple algebra. We need the following lemma.

Lemma 2.6 *Let M be any $\bar{R}_k$-module. Then H is locally finite on M.*

Proof. We first show that M contains a nonzero $\bar{R}_k$-submodule on which H is semisimple. If $AM=0$, then $g_k(H)M=0$ because $[A,B]=g_k(H)$. Since the multiplicity of any root of $g_k(x)=0$ is one, H is semisimple on M. If $AM\neq 0$, let $0\neq u=Av$ for some $v\in M$. Then $(1-2H)u=(1-2H)Av=0$. Thus, $Hu=\frac{1}{2}u$. Then $\bar{R}_k u$ is a nonzero $\bar{R}_k$-submodule on which H semisimple.

Let $F(M)$ be the maximal $\bar{R}_k$-submodule (of M) on which H is locally finite. If $F(M)=M$, we are done. Otherwise, consider the quotient module $W=M/F(M)$. Let W_1 be a nonzero submodule of W on which H is locally finite. Write $W_1=M_1/F(M)$ where $M_1\supset F(M)$ is a submodule of M. Then H is locally finite on M_1. This is a contradiction. $\square$

Theorem 2.7 *$\bar{R}_k$ is semisimple (finite-dimensional) and all inequivalent irreducible $\bar{R}_k$-modules are $L(\frac{n}{2k})$ for $n\in\mathbb{Z}, -(k-1)\leq n\leq k$.*

Proof. To prove that $\bar{R}_k$ is semisimple is equivalent to prove that any $\bar{R}_k$-module is semisimple. Let M be any $\bar{R}_k$-module. Define

$$M_\lambda = \{u\in M \mid (H-\lambda)^n u = 0 \text{ for some positive integer } n.\}$$

Then by Lemma 2.6 we have $M=\oplus_{\lambda\in\mathbb{C}}M_\lambda$. It is easy to see that $AM_\lambda\subseteq M_{\lambda+1}$ and $BM_\lambda\subseteq M_{\lambda-1}$. Let $u\in M_\lambda$ for some $\lambda\in\mathbb{C}$. Use relation $(1-2H)Au=0$ to get $HAu=\frac{1}{2}Au$. Thus $Au\in M_{\frac{1}{2}}$ and $Au=0$ if $\lambda\neq-\frac{1}{2}$. Set $M^1=\oplus_{\lambda\in\frac{1}{2}+\mathbb{Z}}M_\lambda$ and $M^2=\oplus_{\lambda\notin\frac{1}{2}+\mathbb{Z}}M_\lambda$. Then M^1 and M^2 are two submodules of M and $M=M^1\oplus M^2$.

Clearly $AM^2 = 0$. From the proof of Lemma 2.6 we know that all the eigenvalues of H in M^2 are roots of $g_k(x)$. That is,

$$M^2 = \oplus_{n=-k+1}^{k} M_{\frac{n}{2k}}.$$

As $1+\frac{n}{2k}$ is not in $\{\frac{m}{2k} | -k+1 \leq m \leq k\}$ for any $-k+1 \leq n \leq k$ we see that $BM^2 = 0$. So each $M_{\frac{n}{2k}}$ is a direct sum of 1-dimensional irreducible $\bar{R}_k$-modules isomorphic to $L(\frac{n}{2k})$. In fact $L(\frac{n}{2k})$ for $-k+1 \leq n \leq k$ give all the inequivalent 1-dimensional irreducible $\bar{R}_k$-modules.

It remains to show that M^1 is semisimple. Take $\lambda \in \frac{1}{2} + \mathbb{Z}$ with $\lambda \neq \pm\frac{1}{2}$. Then $\lambda - 1 \neq -\frac{1}{2}$ and $AM_\lambda = 0 = ABM_\lambda$. This yields $g_k(H)M_\lambda = 0$. If $M_\lambda \neq 0$, we must have $\lambda = \frac{n}{2k}$ for some $-(k-1) \leq n \leq k-1$. This is a contradiction. Thus $M_\lambda = 0$ if $\lambda \in \frac{1}{2}+\mathbb{Z}, \lambda \neq \pm\frac{1}{2}$ and $M^1 = M_{-\frac{1}{2}} \oplus M_{\frac{1}{2}}$. Consequently, $AM_{\frac{1}{2}} = 0 = BM_{-\frac{1}{2}}$. If $u \in M_{-\frac{1}{2}}$, we get $g_k(-\frac{1}{2})u = g_k(H)u = [A, B]u = BAu$. Since $g_k(-\frac{1}{2}) \neq 0$, A is an injective map from $M_{-\frac{1}{2}}$ to $M_{\frac{1}{2}}$ and B is a surjective map from $M_{\frac{1}{2}}$ to $M_{-\frac{1}{2}}$. Similarly, A is a surjective map from $M_{-\frac{1}{2}}$ to $M_{\frac{1}{2}}$ and B is an injective map. Since H acts on AM as a scalar $\frac{1}{2}$, H acts on $M_{\frac{1}{2}}$ $(= AM_{-\frac{1}{2}})$ as a scalar $\frac{1}{2}$ and H acts on $M_{-\frac{1}{2}}$ $(= BM_{\frac{1}{2}})$ as a scalar $-\frac{1}{2}$. Let u_i for $i \in I$ be a basis of $M_{\frac{1}{2}}$. Then $M^1 = \oplus_{i \in I}(\mathbb{C}u_i + \mathbb{C}Bu_i)$. It is easy to see that for any $i \in I$, $(\mathbb{C}u_i + \mathbb{C}Bu_i)$ is a submodule which is isomorphic to $L(\frac{1}{2})$. Thus M^1 is a direct sum of several copies of 2-dimensional irreducible $\bar{R}_k$-module $L(\frac{1}{2})$. Clearly, $L(\frac{1}{2})$ is the only irreducible $\bar{R}_k$-module whose dimension is greater than 1. $\square$

Remark 2.8 *From the proof of Theorem 2.7, we see that H is semisimple on any $\bar{R}_k$-module M such that any H-weight λ satisfies $2k\lambda \in \mathbb{Z}, -k \leq 2k\lambda \leq k$.*

Let x be an indeterminant. Then we define $\binom{x}{0} = 1$ and $\binom{x}{r} = \frac{1}{r!}x(x-1)\cdots(x-r+1)$ for any positive integer r. Then for any $m \geq n \in \mathbb{Z}_+$, we have:

$$\sum_{i=0}^{n} \binom{n}{i}\binom{x}{m-i} = \binom{x+n}{m}. \tag{2.1}$$

Let L be any positive-definite even lattice and Let $\hat{L}$ be the canonical central extension of L by the cyclic group $\langle \pm 1 \rangle$:

$$1 \to \langle \pm 1 \rangle \to \hat{L} \xrightarrow{-} L \to 1 \tag{2.2}$$

with the commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$. Let $e : \hat{L} \to L$ be a section such that $e_0 = 1$ and $\epsilon : L \times L \to \langle \pm 1 \rangle$ be the corresponding 2-cocycle. Then $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle}$,

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma) \tag{2.3}$$

and $e_\alpha e_\beta = \epsilon(\alpha, \beta)e_{\alpha+\beta}$ for $\alpha, \beta, \gamma \in L$.

Set $\mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. For any $\alpha, \beta \in L$, we define $g_{\alpha,\beta}(x) = 0$ if $\langle \alpha, \beta \rangle \geq 0$, and define

$$g_{\alpha,\beta}(x) = \sum_{r=0}^{\frac{\langle \alpha,\alpha \rangle}{2}-1} \binom{\frac{\langle \alpha,\alpha \rangle}{2} - 1}{r} \binom{x}{-\langle \alpha, \beta \rangle - 1 - r} = \binom{x + \frac{\langle \alpha,\alpha \rangle}{2} - 1}{-\langle \alpha, \beta \rangle - 1} \tag{2.4}$$

if $\langle \alpha, \beta \rangle < 0$.

We define an associative algebra $A(L)$ generated by E_α ($\alpha \in L$) and $\mathbf{h}$ subject to relations:

$$E_0 = 1 \text{ (the identity)}; \tag{2.5}$$

$$hh' - h'h = 0 \quad \text{for any } h, h' \in \mathbf{h}; \tag{2.6}$$

$$hE_\alpha - E_\alpha h = \langle h, \alpha \rangle E_\alpha; \tag{2.7}$$

for any $\alpha, \beta \in L$.

Remark 2.9 *Let L be the root lattice of a semisimple Lie algebra $\mathfrak{g}$ with the set Δ of roots. Then the subalgebra of $A(L)$ generated by $\mathbf{h}$ and E_α for $\alpha \in \Delta$ modulo*

$$E_\alpha E_\beta - E_\beta E_\alpha = E_{\alpha+\beta} g_{\alpha,\beta}(\alpha) \epsilon(\alpha, \beta)$$

is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

Next we define a quotient algebra $\bar{A}(L)$ of $A(L)$ modulo the following relations:

$$\left(\alpha - \frac{\langle \alpha, \alpha \rangle}{2} \right) E_\alpha = 0; \tag{2.8}$$

$$E_\alpha E_\beta = 0 \quad \text{if } \langle \alpha, \beta \rangle > 0; \tag{2.9}$$

$$E_\alpha E_\beta = E_{\alpha+\beta} \epsilon(\alpha, \beta) \binom{\alpha + \frac{\langle \alpha, \alpha \rangle}{2}}{-\langle \alpha, \beta \rangle} \quad \text{if } \langle \alpha, \beta \rangle \leq 0 \tag{2.10}$$

for $\alpha, \beta \in L$.

Proposition 2.10 *Any $\bar{A}(L)$-module is completely reducible. That is, $\bar{A}(L)$ is a (finite-dimensional) semisimple algebra.*

Proof. For any $\alpha \in L$, define $\bar{A}_\alpha(L)$ to be the subalgebra of $\bar{A}(L)$ generated by $\alpha, E_\alpha, E_{-\alpha}$. Note that $\epsilon(\alpha, -\alpha)\epsilon(-\alpha, \alpha) = 1$ and

$$\binom{\alpha + \frac{\langle \alpha, \alpha \rangle}{2}}{\langle \alpha, \alpha \rangle} - \binom{-\alpha + \frac{\langle \alpha, \alpha \rangle}{2}}{\langle \alpha, \alpha \rangle} = \binom{\alpha + \frac{\langle \alpha, \alpha \rangle}{2} - 1}{\langle \alpha, \alpha \rangle - 1}.$$

Set $2k = \langle \alpha, \alpha \rangle$. Then $\bar{A}_\alpha(L)$ is isomorphic to a quotient algebra of $\bar{R}_k$ by sending A, B, H to $\epsilon(\alpha, -\alpha)E_\alpha, E_{-\alpha}, \frac{1}{2k}\alpha$, respectively. Since any $\bar{A}(L)$-module M is a direct sum of irreducible $\bar{A}_\alpha(L)$-modules, α is semisimple on M. Note that L spans $\mathbf{h}$. So $\mathbf{h}$ is semisimple on M. Denote by M_λ the $\mathbf{h}$-weight space of weight λ. Then $M = \oplus_{\lambda \in \mathbf{h}} M_\lambda$. By Remark 2.8 if $M_\lambda \neq 0$ then $|\langle \lambda, \alpha \rangle| \leq \frac{1}{2}\langle \alpha, \alpha \rangle$ and $\langle \lambda, \alpha \rangle \in \mathbb{Q}$ for any $\alpha \in L$.

For any $u \in M_\lambda$, we set $M(u) = \oplus_{\alpha \in L} \mathbb{C} E_\alpha u$. Note that $E_0 = 1$. It follows from the relations (2.7), (2.9) and (2.10) that $M(u)$ is a submodule containing u. Relation (2.8) gives $\alpha E_\alpha u = \frac{1}{2}\langle \alpha, \alpha \rangle E_\alpha u$ for $\alpha \in L$. Use (2.7) to obtain

$$\alpha E_\alpha u = (\langle \alpha, \lambda \rangle + \langle \alpha, \alpha \rangle) E_\alpha u.$$

This gives $(\langle\alpha,\lambda\rangle+\frac{1}{2}\langle\alpha,\alpha\rangle)E_\alpha u=0$. Then either $\langle\alpha,\lambda\rangle+\frac{1}{2}\langle\alpha,\alpha\rangle=0$ or $E_\alpha u=0$. If $\langle\alpha,\lambda\rangle+\frac{1}{2}\langle\alpha,\alpha\rangle\neq 0$, $E_\alpha u$ must be 0. Since L is positive-definite, there are only finitely many $\alpha\in L$ satisfying the relation $\langle\alpha,\lambda\rangle+\frac{1}{2}\langle\alpha,\alpha\rangle=0$. Thus $M(u)$ is finite-dimensional. Suppose $E_\alpha u\neq 0$. Then $\langle\alpha,\lambda\rangle+\frac{1}{2}\langle\alpha,\alpha\rangle=0$ and $E_{-\alpha}E_\alpha u=\epsilon(-\alpha,\alpha)u$ (see definition (2.10)). This proves that $M(u)$ is a finite-dimensional irreducible $\bar{A}(L)$-module. As a result, M is a direct sum of finite-dimensional irreducible $\bar{A}(L)$-modules. This completes the proof. $\square$

Remark 2.11 *Let M be an $\bar{A}(L)$-module. Then it follows from the proof of Proposition 2.10 and Remark 2.9 that any $\mathbf{h}$-weight λ of M is in the dual lattice $L^\circ=\{\lambda\in\mathbf{h}|\langle L,\lambda\rangle\subset\mathbb{Z}\}$ of L and satisfies the relation: $|\langle\lambda,\alpha\rangle|\leq\frac{1}{2}\langle\alpha,\alpha\rangle$ for any $\alpha\in L$, which is equivalent to the relation: $\langle\lambda+\alpha,\lambda+\alpha\rangle\geq\langle\lambda,\lambda\rangle$ for any $\alpha\in L$.*

Next, we construct all irreducible $\bar{A}(L)$-modules. Set

$$S=\{\lambda\in L^\circ|\langle\lambda+\alpha,\lambda+\alpha\rangle\geq\langle\lambda,\lambda\rangle\quad\text{for any }\alpha\in L\}.$$

For any $\lambda\in S$, we define

$$\Delta(\lambda)=\{\alpha\in L|\langle\lambda+\alpha,\lambda+\alpha\rangle=\langle\lambda,\lambda\rangle\}.$$

Then for any $\lambda\in S,\alpha\in\Delta(\lambda)$, we have $\lambda+\alpha\in S$ because

$$\langle\lambda+\alpha+\beta,\lambda+\alpha+\beta\rangle\geq\langle\lambda,\lambda\rangle=\langle\lambda+\alpha,\lambda+\alpha\rangle$$

for any $\beta\in L$.

For any $\lambda\in S$, let M^λ be a vector space with a basis $\{u_\alpha^\lambda|\alpha\in\Delta(\lambda)\}$. Define

$$hu_\alpha^\lambda=\langle\lambda+\alpha,h\rangle u_\alpha^\lambda\quad\text{for }h\in\mathbf{h},\alpha\in\Delta(\lambda).$$

For $\beta\in L,\alpha\in\Delta(\lambda)$, we define $E_\beta u_\alpha^\lambda=\epsilon(\alpha,\beta)u_{\alpha+\beta}^\lambda$ if $\beta\in\Delta(\alpha+\lambda)$ and $E_\beta u_\alpha^\lambda=0$ otherwise.

Proposition 2.12 *The following hold:*

(1) The vector space M^λ together with the defined action is an irreducible $\bar{A}(L)$-module for any $\lambda\in S$.

(2) $M^{\lambda_1}\simeq M^{\lambda_2}$ if and only if $\lambda_2=\lambda_1+\alpha$ for some $\alpha\in\Delta(\lambda_1)$.

(3) Any irreducible $\bar{A}(L)$-module is isomorphic to M^λ for some $\lambda\in S$.

Proof. For (1) we need to establish the defining relations (2.5)-(2.10) of $\bar{A}(L)$. Relations (2.5)-(2.7) are clear. Let $\alpha\in\Delta(\lambda)$ and $\beta\in L$. If $\beta\in\Delta(\lambda+\alpha)$ then $2\langle\lambda+\alpha,\beta\rangle+\langle\beta,\beta\rangle=0$. Thus

$$(\beta-\frac{\langle\beta,\beta\rangle}{2})E_\beta u_\alpha^\lambda=(\beta-\frac{\langle\beta,\beta\rangle}{2})\epsilon(\beta,\alpha)u_{\alpha+\beta}^\lambda=(\langle\lambda+\alpha+\beta,\beta\rangle-\frac{\langle\beta,\beta\rangle}{2})\epsilon(\beta,\alpha)u_{\alpha+\beta}^\lambda=0.$$

If $\beta \notin \Delta(\lambda+\alpha)$, $E_\beta u^\lambda_\alpha = 0$. This gives relation (2.8).

We now show realtion (2.9). Let $\beta_1, \beta_2 \in L$ such that $\langle \beta_1, \beta_2 \rangle > 0$. We have to show that $E_{b_1} E_{b_2} u^\lambda_\alpha = 0$ for any $\alpha \in \Delta(\lambda)$, or equivalently, either $\beta_2 \notin \Delta(\alpha+\lambda)$ or $\beta_1 \notin \Delta(\alpha+\beta_1+\lambda)$. If $\beta_2 \in \Delta(\alpha+\lambda)$ and $\beta_1 \in \Delta(\alpha+\beta_2+\lambda)$ then

$$\langle \lambda+\alpha+\beta_1+\beta_2, \lambda+\alpha+\beta_1+\beta_2 \rangle = \langle \lambda+\alpha+\beta_2, \lambda+\alpha+\beta_2 \rangle = \langle \lambda+\alpha, \lambda+\alpha \rangle = \langle \lambda, \lambda \rangle$$

and $\beta_1 \in \Delta(\alpha+\beta_2+\lambda)$. This implies that

$$\langle \lambda+\alpha, \beta_1 \rangle + \frac{1}{2}\langle \beta_1, \beta_1 \rangle + \langle \beta_1, \beta_2 \rangle = 0.$$

Since $\langle \lambda+\alpha, \beta_1 \rangle + \frac{1}{2}\langle \beta_1, \beta_1 \rangle \geq 0$ we see that $\langle \beta_1, \beta_2 \rangle \leq 0$. This is a contradiction.

It remains to show (2.10). Let $\beta_1, \beta_2 \in L$ such that $\langle \beta_1, \beta_2 \rangle \leq 0$. Take $\alpha \in \Delta(\lambda)$. There are 3 cases:

Case 1: $\beta_2, \beta_1+\beta_2 \in \Delta(\alpha+\lambda)$. From the previous paragraph, $\beta_1 \in \Delta(\alpha+\beta_2+\lambda)$. Use (2.3) to obtain

$$\begin{aligned} E_{\beta_1} E_{\beta_2} u^\lambda_\alpha &= \epsilon(\beta_1, \beta_2+\alpha)\epsilon(\beta_2, \alpha) u^\lambda_{\beta_1+\beta_2+\alpha} \\ &= \epsilon(\beta_1, \beta_2)\epsilon(\beta_1+\beta_2, \alpha) u^\lambda_{\beta_1+\beta_2+\alpha} \\ &= \epsilon(\beta_1, \beta_2) E_{\beta_1+\beta_2} u^\lambda_\alpha. \end{aligned}$$

Also note that $\langle \beta_1, \beta_2 \rangle \leq 0$. So $\binom{\langle \beta_1, \alpha+\lambda \rangle + \frac{\langle \beta_1, \beta_1 \rangle}{2}}{-\langle \beta_1, \beta_2 \rangle} = 1$. (2.10) is true in this case.

Case 2: $\beta_1+\beta_2 \notin \Delta(\alpha+\lambda)$. Then $E_{\beta_1+\beta_2} u^\lambda_\alpha = 0$. From the discussion before we see that either $\beta_2 \notin \Delta(\alpha+\lambda)$ or $\beta_1 \notin \Delta(\alpha+\beta_1+\lambda)$. So $E_{\beta_1} E_{\beta_2} u^\lambda_\alpha = 0$.

Case 3: $\beta_2 \notin \Delta(\alpha+\lambda)$ and $\beta_1+\beta_2 \in \Delta(\alpha+\lambda)$. In this case $E_{\beta_1} E_{\beta_2} u^\lambda_\alpha = 0$, $\langle \lambda+\alpha, \beta_2 \rangle + \frac{1}{2}\langle \beta_2, \beta_2 \rangle > 0$ and

$$\langle \lambda+\alpha, \beta_1+\beta_2 \rangle + \frac{1}{2}\langle \beta_1+\beta_2, \beta_1+\beta_2 \rangle = 0.$$

we have

$$0 \leq \langle \lambda+\alpha, \beta_1 \rangle + \frac{1}{2}\langle \beta_1, \beta_1 \rangle < -\langle \beta_1, \beta_2 \rangle.$$

Then $\binom{\langle \lambda+\alpha, \beta_1 \rangle + \frac{1}{2}\langle \beta_1, \beta_1 \rangle}{-\langle \beta_1, \beta_2 \rangle} = 0$, as desired.

It is clear that M^λ is irreducible.

(2) follows from the definition of M^λ and (3) follows from the proof of Proposition 2.10 immediately. $\square$

Corollary 2.13 *There is a one-to-one correspondence between set of equivalence classes of irreducible $\bar{A}(L)$-modules and the set of cosets of L in L°.*

Proof. For any $\lambda_1, \lambda_2 \in S$, we define $\lambda_1 \equiv \lambda_2$ if $\lambda_2 = \lambda_1 + \alpha$ for some $\alpha \in \Delta(\lambda_1)$. If $\lambda_1, \lambda_2 \in S$ satisfies $\lambda_2 - \lambda_1 = \alpha \in L$, then it follows from the definitions of S and $\Delta(\lambda_1)$ that $\alpha \in \Delta(\lambda_1)$. Then we have an equivalent relation on S, which is the restriction of the congruence relation of L° modulo L. On the other hand, for any coset $\lambda + L \in L^\circ/L$, since L is positive-definite, there is an element $\beta \in \lambda + L$ (as a set) with a minimal norm. Clearly $\beta \in S$. The corollary follows from Proposition 2.12. $\square$

3 Zhu's algebra $A(V_L)$

First we recall from [FLM] the explicit construction of vertex operator algebra V_L. Let L be a positive-definite even lattice. Set $\mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-form on L to $\mathbf{h}$. Let $\hat{\mathbf{h}} = \mathbf{C}[t, t^{-1}] \otimes \mathbf{h} \oplus \mathbf{C}c$ be the affinization of $\mathbf{h}$, i.e., $\hat{\mathbf{h}}$ is a Lie algebra with commutator relations:

$$[t^m \otimes h, t^n \otimes h'] = m\delta_{m+n,0}\langle h, h'\rangle c \text{ for } h, h' \in \mathbf{h}; m, n \in \mathbb{Z};$$

$$[\hat{\mathbf{h}}, c] = 0.$$

We also use the notation $h(n) = t^n \otimes h$ for $h \in \mathbf{h}, n \in \mathbb{Z}$.

Set

$$\hat{\mathbf{h}}^+ = t\mathbf{C}[t] \otimes \mathbf{h}; \;\; \hat{\mathbf{h}}^- = t^{-1}\mathbf{C}[t^{-1}] \otimes \mathbf{h}.$$

Then $\hat{\mathbf{h}}^+$ and $\hat{\mathbf{h}}^-$ are abelian subalgebras of $\hat{\mathbf{h}}$. Let $U(\hat{\mathbf{h}}^-) = S(\hat{\mathbf{h}}^-)$ be the universal enveloping algebra of $\hat{\mathbf{h}}^-$. Consider the induced $\hat{\mathbf{h}}$-module

$$M(1) = U(\hat{\mathbf{h}}) \otimes_{U(\mathbf{C}[t]\otimes\mathbf{h}\oplus\mathbf{C}c)} \mathbf{C} \simeq S(\hat{\mathbf{h}}^-) \text{ (linearly)},$$

where $\mathbf{C}[t] \otimes \mathbf{h}$ acts trivially on $\mathbf{C}$ and c acts on $\mathbf{C}$ as multiplication by 1.

Recall from (2.2) that $\hat{L}$ be the canonical central extension of L by the cyclic group $\langle \pm 1\rangle$. Form the induced $\hat{L}$-module

$$\mathbb{C}\{L\} = \mathbf{C}[\hat{L}] \otimes_{\langle \pm 1\rangle} \mathbf{C} \simeq \mathbf{C}[L] \text{ (linearly)},$$

where $\mathbf{C}[\cdot]$ denotes the group algebra and -1 acts on $\mathbf{C}$ as multiplication by -1. For $a \in \hat{L}$, write $\iota(a)$ for $a \otimes 1$ in $\mathbf{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by: $a \cdot \iota(b) = \iota(ab)$ and $(-1) \cdot \iota(b) = -\iota(b)$ for $a, b \in \hat{L}$.

Furthermore we define an action of $\mathbf{h}$ on $\mathbf{C}\{L\}$ by: $h \cdot \iota(a) = \langle h, \bar{a}\rangle \iota(a)$ for $h \in \mathbf{h}, a \in \hat{L}$. Define $z^h \cdot \iota(a) = z^{\langle h, \bar{a}\rangle}\iota(a)$.

The untwisted space associated with L is defined to be

$$V_L = \mathbf{C}\{L\} \otimes_{\mathbf{C}} M(1) \simeq \mathbf{C}[L] \otimes S(\hat{\mathbf{h}}^-) \text{ (linearly)}.$$

Then $\hat{L}, \hat{\mathbf{h}}, z^h$ $(h \in \mathbf{h})$ act naturally on V_L by acting on either $\mathbf{C}\{L\}$ or $M(1)$ as indicated above.

For $h \in \mathbf{h}$ set

$$h(z) = \sum_{n\in\mathbb{Z}} h(n) z^{-n-1}.$$

We use a normal ordering procedure, indicated by open colons, which signify that in the enclosed expression, all creation operators $h(n)$ $(n < 0)$,$a \in \hat{L}$ are to be placed to the left of all annihilation operators $h(n), z^h$ $(h \in \mathbf{h}, n \geq 0)$. For $a \in \hat{L}$, set

$$Y(\iota(a), z) = {}^\circ_\circ e^{\int(\bar{a}(z) - \bar{a}(0)z^{-1})} a z^{\bar{a}} {}^\circ_\circ .$$

Let $a \in \hat{L};\ h_1, \cdots, h_k \in \mathbf{h}; n_1, \cdots, n_k \in \mathbb{Z}\ (n_i > 0)$. Set

$$v = \iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k) \in V_L.$$

Define vertex operator

$$Y(v,z) = {\circ \atop \circ} \left(\frac{1}{(n_1-1)!} (\frac{d}{dz})^{n_1-1} h_1(z) \right) \cdots \left(\frac{1}{(n_k-1)!} (\frac{d}{dz})^{n_k-1} h_k(z) \right) Y(\iota(a), z) {\circ \atop \circ}$$

This gives us a well-defined linear map

$$\begin{aligned} Y(\cdot, z): \quad & V_L \to (\mathrm{End} V_L)[[z, z^{-1}]] \\ & v \mapsto Y(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1},\ (v_n \in \mathrm{End} V_L). \end{aligned}$$

Let $\{\ \alpha_i \mid i = 1, \cdots, d\}$ be an orthonormal basis of $\mathbf{h}$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^{d} \alpha_i(-1)\alpha_i(-1) \in V_L.$$

Then $Y(\omega, z) = \sum_{n \in \mathbf{Z}} L(n) z^{-n-2}$ gives rise to a representation of the Virasoro algebra on V_L and

$$\begin{aligned} & L(0)\left(\iota(a) \otimes h_1(-n_1) \cdots h_n(-n_k)\right) \\ = \quad & \left(\frac{1}{2}\langle \bar{a}, \bar{a} \rangle + n_1 + \cdots + n_k\right) \left(\iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k)\right). \end{aligned} \tag{3.1}$$

The following theorem was due to Borcherds [B] and Frenkel, Lepowsky and Meurman [FLM].

Theorem 3.1 $(V_L, Y, \mathbf{1}, \omega)$ *is a vertex operator algebra.*

Recall the dual lattice L° of L from Section 2. Let $\{\beta_i + L | i = 1, 2, \cdots, n\}$ be a full set of representives of cosets of L in L°. Then $V_{\beta_i + L}$ is an irreducible V_L-module for each i [FLM] and it was also proved in [D] that these are all irreducible V_L-modules up to equivalence; see also [DLM1].

Define the Schur polynomials $p_r(x_1, x_2, \cdots)$ $(r \in \mathbb{Z}_+)$ in variables $x_1, x_2, \cdots$ by the following equation:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{r=0}^{\infty} p_r(x_1, x_2, \cdots) y^r. \tag{3.2}$$

For any monomial $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ we have an element $h(-1)^{n_1} h(-2)^{n_2} \cdots h(-r)^{n_r} \mathbf{1}$ in V_L for $h \in \mathbf{h}$. Then for any polynomial $f(x_1, x_2, \cdots)$, $f(h(-1), h(-2), \cdots)\mathbf{1}$ is a well-defined element in V_L. In particular, $p_r(h(-1), h(-2), \cdots)\mathbf{1}$ for $r \in \mathbb{Z}_+$ are elements of V_L.

Suppose $a, b \in \hat{L}$ such that $\bar{a} = \alpha, \bar{b} = \beta$. Then

$$\begin{aligned} Y(\iota(a), z)\iota(b) &= z^{\langle\alpha,\beta\rangle} \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(-n)}{n} z^n\right) \iota(ab) \\ &= \sum_{r=0}^{\infty} p_r(\alpha(-1), \alpha(-2), \cdots)\iota(ab) z^{r+\langle\alpha,\beta\rangle}. \end{aligned} \tag{3.3}$$

Thus

$$\iota(a)_i \iota(b) = 0 \quad \text{for } i \geq -\langle\alpha, \beta\rangle. \tag{3.4}$$

Especially, if $\langle\alpha, \beta\rangle \geq 0$, we have $\iota(a)_i\iota(b) = 0$ for all $i \in \mathbb{Z}_+$, and if $\langle\alpha, \beta\rangle = -n < 0$, we get

$$\iota(a)_{i-1}\iota(b) = p_{n-i}(\alpha(-1), \alpha(-2), \cdots)\iota(ab) \quad \text{for } i \in \mathbb{Z}_+. \tag{3.5}$$

It is well-known that V_L is generated by u_n for $u \in \{\iota(a), \alpha(-1) | a \in \hat{L}\}, n \in \mathbb{Z}$.

In [Z], an associative algebra $A(V)$ was introduced for any vertex operator algebra $V = \oplus_{n\in\mathbb{Z}} V_n$ such that there is a 1-1 correspondence between the set of equivalence classes of irreducible V-modules (without assuming that homogeneous subspaces are finite-dimensional) and the set of all equivalence classes of irreducible $A(V)$-modules on which the central element (obtained from the Virasoro element ω) as a scalar.

Here is the definition of $A(V)$. Define two bilinear products $*$ and $\circ$ on V as follows:

$$u * v = \mathrm{Res}_z \frac{(1+z)^n}{z} Y(u, z)v = \sum_{i=0}^{\infty} \binom{n}{i} u_{i-1}v;$$

$$u \circ v = \mathrm{Res}_z \frac{(1+z)^n}{z^2} Y(u, z)v = \sum_{i=0}^{\infty} \binom{n}{i} u_{i-2}v$$

for any $u \in V_n, v \in V$. Denote by $O(V)$ the linear span of all $u \circ v$ for $u, v \in V$ and set $A(V) = V/O(V)$. Then $A(V)$ is an associative algebra under $*$ with identity $\mathbf{1} + O(V)$. For any (weak) V-module M, we define [DLM2]

$$\Omega(M) = \{u \in M | a_m u = 0 \quad \text{for any } a \in V_n, m > n-1\}.$$

Then $\Omega(M)$ is a natural $A(V)$-module with $a + O(V)$ acting as $o(a) = a_{n-1}$ for $a \in V_n$. This gives is a 1-1 correspondence between the set of equivalence classes of irreducible V-modules (without assuming that homogeneous subspaces are finite-dimensional) and the set of all equivalence classes of irreducible $A(V)$-modules [Z].

The following relations from [Z] will be useful later:

$$u * v \equiv \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}v-1}}{z} Y(v, z)u \equiv \sum_{i=0}^{\infty} \binom{\mathrm{wt}v - 1}{i} v_{i-1}v \mod O(V); \tag{3.6}$$

$$u * v - v * u \equiv \mathrm{Res}_z (1+z)^{\mathrm{wt}u-1} Y(u, z)v \equiv \sum_{i=0}^{\infty} \binom{\mathrm{wt}u - 1}{i} u_i v \mod O(V). \tag{3.7}$$

Theorem 3.2 *Let $L = \mathbb{Z}\alpha$ be a one-dimensional lattice with $|\alpha|^2 = 2k$ for $k \in \mathbb{N}$ and let V_L be the vertex operator algebra associated with L. Then Zhu's algebra $A(V_L)$ is isomorphic to $\bar{R}_k = R_k/\langle (1-2H)A\rangle$.*

Proof. In this case $\hat{L}$ is a direct product of L and $\langle \pm 1\rangle$ and we regard L as a subgroup of $\hat{L}$. The section e is the identity map and the cocycle $\epsilon(\cdot,\cdot)$ is trivial. Let $a \in \hat{L}$ such that $\bar{a} = \alpha$. Set $e = \iota(a), f = \iota(a^{-1}), h = \alpha(-1)\mathbf{1}$. From the construction of V_L and (3.5) we have

$$\begin{aligned} &h_i e = 2k\delta_{i,0}e, \quad h_i f = -2k\delta_{i,0}f, \ h_i h = 2k\delta_{i,1}\mathbf{1};\\ &e_i f = p_{2k-1-i}(h_{-1}, h_{-2}, \cdots)\mathbf{1} \end{aligned}$$

for any $i \in \mathbb{Z}_+$. Since $\mathrm{wt}h = 1, \mathrm{wt}e = \mathrm{wt}f = k$, by (3.7) we have

$$\begin{aligned} h*e - e*h &\equiv \mathrm{Res}_z \frac{1}{z} Y(h,z)e \equiv h_0 e \equiv 2ke \mod O(V_L);\\ h*f - f*h &\equiv \mathrm{Res}_z \frac{1}{z} Y(h,z)f \equiv h_0 f \equiv -2kf \mod O(V_L);\\ e*f - f*e &\equiv \sum_{i=0}^{k-1} \binom{k-1}{i} e_i f\\ &\equiv \sum_{i=0}^{k-1} \binom{k-1}{i} p_{2k-1-i}(h(-1), h(-2), \cdots)\mathbf{1} \mod O(V_L). \end{aligned}$$

For any $r \geq 1$ and $u \in V_L$ we have

$$(h(-r-1) + h(-r))u = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}h}}{z^{r+1}} u \in O(V)$$

(cf. [FZ]). Thus $h(-r-1) \equiv -h(-r)u \mod O(V)$ for any $u \in V_L$. Then for any $r \in \mathbb{Z}_+, n_1, \cdots, n_r \in \mathbb{N}$, we have:

$$h(-n_1)\cdots h(-n_r)\mathbf{1} + O(V_L) = (-1)^{n_1+\cdots+n_r+r} h^r + O(V_L). \tag{3.8}$$

Let $\bar{p}_r(x) = p_r(x, -x, x, \cdots)$, *i.e.*, substitute x_n by $(-1)^{n-1}x$ for any $n \geq 1$. Since

$$\exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x}{n} y^n\right) = (1+y)^x = \sum_{r\geq 0} \binom{x}{r} y^r$$

we see that $\bar{p}_r(x) = \binom{x}{r} = \frac{1}{r!}x(x-1)\cdots(x+1-r)$ for $r \in \mathbb{Z}_+$. Hence

$$\begin{aligned} &e*f - f*e \equiv \sum_{i=0}^{k-1} \binom{k-1}{i} \bar{p}_{2k-1-i}(h) \equiv \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{h}{2k-1-i}\\ &\equiv \binom{h+k-1}{2k-1} \mod O(V_L). \end{aligned}$$

Here $\binom{h+k-1}{2k-1}$ is understood to be $\frac{1}{(2k-1)!}(h+k-1)*(h+k-2)*\cdots*(h-k+1)$ and this consideration also applies in the next theorem. This gives an algebra homomorphism ψ from R_k to $A(V_L)$ such that $\psi(A)=e+O(V_L)$, $\psi(B)=f+O(V_L)$, $\psi(H)=\frac{1}{2k}h+O(V_L)$.

Set $W_\pm=\sum_{n=0}^{\infty}e^{\pm n\alpha}\otimes M(1)$. Then $V_L=W_-+W_+$. Consider $G_+=\sum_{m\in\mathbb{Z}}\mathbb{C}e_m$ and $G_-=\sum_{m\in\mathbb{Z}}\mathbb{C}f_m$. Then $G_\pm$ are abelian Lie subalgberas of $G_\pm+\hat{\mathbf{h}}$ which is a Lie subalgebra of EndV_L. It is clear that $W_\pm$ is generated by $G_\pm+\hat{\mathbf{h}}$ from the vacuum vector, i.e., $W_\pm=U(G_\pm+\hat{\mathbf{h}})\mathbf{1}$. By PBW theorem we have $W_\pm=U(G_\pm+\hat{\mathbf{h}})\mathbf{1}=U(\hat{\mathbf{h}}^-)U(G_\pm)\mathbf{1}$. Since $e_m\mathbf{1}=0$ for $m\in\mathbb{Z}_+$, W_+ can be generated from the vacuum $\mathbf{1}$ by the following operators:

$$\mathrm{Res}_z\frac{(1+z)^k}{z^{n+1}}Y(e,z)=\sum_{i=0}^{k}\binom{k}{i}e_{-n-1+i};$$
$$\mathrm{Res}_z\frac{(1+z)^1}{z^{n+1}}Y(h,z)=h(-n-1)+h(-n)$$

for $n\in\mathbb{Z}_+$. Similarly for W_-. This implies that the map ψ is onto.

A straightforward calculation gives $L(-1)\iota(a)=\alpha(-1)\iota(a)$. Then

$$L(-1)\iota(a)=(\alpha(-1)+\alpha(0))\iota(a)-\alpha(0)\iota(a)=(\alpha(-1)+\alpha(0))\iota(\alpha)-2k\iota(a). \qquad (3.9)$$

Since $L(0)+L(-1)$ maps V to $O(V)$ (see [Z]) and $h*\iota(a)=(\alpha(-1)+\alpha(0))\iota(a)$ we have $-k\iota(a)=-L(0)\iota(a)\equiv L(-1)v=h*\iota(a)-2k\iota(a)$. This gives rise to the relation $-k\psi(A)=2k\psi(H)*\psi(A)-2k\psi(A)$ in $A(V_L)$. That is, $(1-2\psi(H))*\psi(A)=0$. This shows that $A(V_L)$ is a quotient algebra of $\bar{R}_k$. Since V_L already has irreducible modules $V_{L+\frac{n}{2k}\alpha}$ for $-(k-1)\le n\le k$ and $\bar{R}_k$ has exactly $2k$ irreducible modules, $A(V_L)$ must be isomorphic to $\bar{R}_k$. $\square$

Remark 3.3 *It is interesting to know if one can construct a vertex operator algebra whose Zhu's algebra is exactly R_k. Notice that $\{e_n,f_n,h_n|n\in\mathbb{Z}\}$ generates a topological associative algebra A because of the infinite sum relation (3.3), which is not a linear Lie algebra. In order to construct such a vertex operator algebra one may need to develop the notion of generalized Verma modules for A and establish some results on generalized Verma modules. Of course, if we took these for granted, then we could have a vertex operator algebra with R_k as its Zhu's algebra.*

Theorem 3.4 *Let L be any positive-definite even lattice. Then Zhu's algebra $A(V_L)$ is isomorphic to $\bar{A}(L)$.*

Proof. The proof is similar to that of Theorem 3.2. First, we establish an algebra homomorphism ψ from $\bar{A}(L)$ onto $A(V_L)$. Recall from Section 2 the section $e:L\to\hat{L}$. Define a linear map ψ from $\bar{A}(L)$ to V_L as follows:

$$\psi(E_\alpha)=\iota(e_\alpha),\psi(h)=h_{-1}\mathbf{1}=h(-1)$$

for $\alpha \in L, h \in \mathbf{h}$. From (3.7) we have

$$h(-1) * h'(-1) - h'(-1) * h(-1) \equiv \mathrm{Res}_z \frac{1}{z} Y(h(-1), z) h'(-1) + O(V) = 0$$

for $h, h' \in \mathbf{h}$. Similarly, for $a \in \hat{L}, h \in \mathbf{h}$ with $\bar{a} = \alpha$

$$h(-1) * \iota(a) - \iota(a) * h(-1) \equiv h_0 \iota(a) \equiv \langle h, \alpha \rangle \iota(a) \mod O(V_L).$$

The same calculation in (3.9) gives

$$L(-1)\iota(a) = \alpha(-1) * \iota(a) - \langle \alpha, \alpha \rangle \iota(a).$$

Thus $L(-1)\iota(a) + L(0)\iota(a) = \alpha(-1) * \iota(a) - \frac{\langle \alpha, \alpha \rangle}{2} \iota(a) \in O(V_L)$. This shows that relations (2.5)-(2.8) hold.

Let $a, b \in \hat{L}$ such that $\bar{a} = \alpha$ and $\bar{b} = \beta$. Then

$$\iota(a) * \iota(b) = \sum_{i=0}^{m} \binom{m-1}{i} \iota(a)_{i-1} \iota(b)$$

where $m = \frac{\langle \alpha, \alpha \rangle}{2}$ is the weight of $\iota(a)$. If $\langle \alpha, \beta \rangle > 0$, by (3.4) we have $\iota(\alpha)_i \iota(\beta) = 0$ for all $i \geq -1$, so that $\iota(\alpha) * \iota(\beta) = 0$ and (2.9) holds.

If $\langle \alpha, \beta \rangle = -n \leq 0$, by (3.5) we have

$$\iota(a) * \iota(b) = \sum_{i=0}^{m} \binom{m}{i} p_{n-i}(\alpha(-1), \alpha(-2), \cdots) \iota(ab).$$

Now take $a = e_\alpha$ and $b = e_\beta$. As in (3.8) we obtain

$$\begin{aligned} & \iota(a) * \iota(b) + O(V_L) \\ = & \sum_{i=0}^{m} \binom{m}{i} p_{n-i}(\alpha(-1), -\alpha(-1), \cdots) \iota(ab) + O(V_L). \end{aligned}$$

Since $\mathrm{wt} h(-1) = 1$, by (3.6) we get $u * h(-1) + O(V_L) = h(-1)u + O(V_L)$ for any $u \in V_L$. Then $h(-1)^k \iota(a) + O(V_L) = u * h(-1)^k$ for any nonnegative integer k. As in the previous theorem $h(-1)^k$ is understood to be the multiplication $*$ k times. Thus

$$\begin{aligned} & \iota(a) * \iota(b) + O(V_L) \\ = & \sum_{i=0}^{m} \binom{m}{i} p_{n-i}(\alpha(-1), -\alpha(-1), \cdots) \iota(ab) + O(V_L) \\ = & \iota(ab) * \left(\sum_{i=0}^{m} \binom{m}{i} p_{n-i}(\alpha(-1), -\alpha(-1), \cdots) \right) + O(V_L) \\ = & \iota(ab) * \left(\sum_{i=0}^{m} \binom{m}{i} \binom{\alpha(-1)}{n-i} \right) + O(V_L) \\ = & \iota(ab) * \binom{\alpha(-1) + m}{n} + O(V_L). \end{aligned}$$

So the relation (2.10) is true and ψ is an algebra homomorphism from $\bar{A}(L)$ into $A(V_L)$.

Recall that $\{\alpha_1, ..., \alpha_d\}$ is an orthonomal basis for $\mathbf{h}$. Let $u = p(\alpha_i(-j))\iota(a)$ for $a \in \hat{L}$ and $p(x_{i,j}) \in \mathbb{C}[x_{i,j} | 1 \le i \le d, j = 1, 2, \cdots]$. Then from the previous paragraph we see that

$$u \equiv p((-1)^{j-1}\alpha_i(-1))\iota(a) \equiv \iota(a) * (p(\alpha_i(-1))\mathbf{1}) \mod O(V). \tag{3.10}$$

Since such u span V_L (by the construction of V_L), ψ is onto. Since V_L already has $|L^\circ/L|$ modules (see [FLM] and [D]), it follows from Proposition 2.10 that ψ is an isomorphism. $\square$

Remark 3.5 *Since there is a 1-1 correspondence between the set of equivalence classes of irreducible V_L-modules and the set of equivalence classes of irreducible $A(V_L)$-modules, our result on $A(V_L)$ gives an alternative approach to the classification of irreducible V_L-modules obtained in [D].*

References

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068-3071.

[BB] A. Beilinson and J. N. Bernstein, Localisation de $\mathfrak{g}$-modules, *C. R. Acade. Sci. Ser.* **A-B 292** (1981), 15-18.

[BD] A. Beilinson and V. Drinfeld, unpublished manuscript.

[D] C. Dong, vertex algebras associated to even lattices, *J. Alg.* **160** (1993), 245-265.

[DLM1] C. Dong, H. Li and G. Mason, Regularity of rational vertex operator algebras, q-alg/9508018.

[DLM2] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, q-alg/9509005.

[DLM3] C. Dong, H. Li and G. Mason, Vertex operator algebras associated to admissible representations of $\hat{s}l_2$, q-alg/9509026.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs American Math. Soc.* **104**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster, Pure and Applied Math.,* **Vol. 134**, Academic Press, New York, 1988.

[FZ] I. B. Frenkel and Y.-C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.

[H] T. J. Hodges, Noncommutative deformations of Type-A Kleinian singularities, *J. Alg.* **161** (1993), 271-290.

[HL] Yi-Zhi Huang and J. Lepowsky, On the $\mathcal{D}$-module and formal-variable approaches to vertex algebras, q-alg/9603020.

[HS] T. J. Hodges and S. P. Smith, Sheaves of noncommutative algebras and the Beilinson-Bernstein equivalence of categories, *Proc. Amer. Math. Soci.* **93** (1985), 379-386.

[K] V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[Li] H.-S. Li, Local systems of vertex operators, vertex superalgebras and modules, *J. of Pure and Appl. Alg.*, to appear.

[Lu] Klaus Lucke, On Zhu's associative algebra as a tool in the representation theory of vertex operator algebras, hep-th/9605020.

[S] S. P. Smith, A class of algebras similar to the enveloping algebra of sl_2, *Trans. Amer. Math. Soci.* **322** (1990), 285-314.

[Z] Y.-C. Zhu, Vertex operator algebras, elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.